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On the Ground State of Quantum Gravity

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Abstract

In order to gain insight into the possible Ground State of Quantized Einstein's Gravity, we have devised a variational calculation of the energy of the quantum gravitational field in an open space, as measured by an asymptotic observer living in an asymptotically flat space-time. We find that for Quantum Gravity (QG) it is energetically favourable to perform its quantum fluctuations not upon flat space-time but around a "gas" of wormholes, whose size is the Planck length a_p ($a_p \simeq 10^{-33}$cm). As a result, assuming such configuration to be a good approximation to the true Ground State of Quantum Gravity, space-time, the arena of physical reality, turns out to be well described by Wheeler's Quantum Foam and adequately modeled by a space-time lattice with lattice constant a_p, the Planck lattice.

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1 Introduction

Among the fundamental interactions of Nature, since the monumental contribution of Albert Einstein, Gravity plays the central role of determining the structure of space-time, the arena of physical reality. As well known, in classical physics a world without matter, the Vacuum, has the simplest of all structures, it is flat (pseudoeuclidean); but in quantum physics? This is the central question that has occupied the best theoretical minds since it became apparent, at the beginning of the 30's, that Quantum Field Theory (QFT) is the indispensable intellectual tool for discovering the extremely subtle ways in which the quantum world actually works. Thus the problem to solve was to find in some way or other the Ground State (GS) of Quantum Gravity (QG), which determines the dynamical behaviour of any physical system, through the non-trivial structure that space-time acquires as a result of the quantum fluctuations that in such state the gravitational field, like all quantum fields, must experience. Of course this problem, at least in the non-perturbative regime, is a formidable one, and many physicists, J.A. Wheeler foremost among them, could but speculate about the ways in which the expected violent quantum fluctuations at the Planck distance a_p ($a_p \simeq 10^{-33} cm$) could change the space-time structure of the Vacuum from its classical, trivial (pseudoeuclidean) one. And Wheeler's conjecture [1], most imaginative and intriguing, of a space-time foam vividly expresses the intuition that at the Planck distance the fluctuations of the true QG ground state would end up in submitting the classical continuum of events to a metamorphosis into an essentially discontinuous, discrete structure.[1]

It is the purpose of this letter to report on the recent results of an investigation on a possible QG ground state. The starting point of our attack is the realization that QG can be looked at as a non-abelian gauge theory whose gauge group is the Poincaré group. Following the analysis performed by one of us (GP) [3] of another non-abelian gauge theory QCD (whose gauge group is $SU_c(3)$), we decided to explore the possibility that the energy density (to be appropriately defined, see below) of the quantum fluctuations of the gravitational field around a non-trivial classical solution of Einstein's field equations for the matterless world, could be lower than the energy of the perturbative ground state (PGS), which comprises the zero point fluctuations of the gravitational field's modes around flat space-time. Indeed in QCD it was found that the unstable modes (imaginary frequencies) of the gauge fields around the classical constant chromomagnetic field, solution of the empty space Yang-Mills equations, in the average screen completely the classical chromomagnetic field, allowing the interaction energy between such field and the short wave-length fluctuations of the quantized gauge

[1] We should like to recall here that, based on Wheeler's idea, a successful research program was initiated a few years ago to explore the consequences of the Standard Model ($SU_c(3) \otimes SU_L(2) \otimes U_Y(1)$) in a discrete space-time, conveniently modeled by a lattice of constant a_p, the Planck lattice (PL). For a recent review one may consult ref.[2].

field to lower the energy density of such configuration below the PGS energy density. Thus we decided to try for QG the strategy that was successful in QCD, i.e.

1. select a class of empty space classical solutions of Einstein's equations that is simple and manageable;
2. evaluate the spectrum of the small amplitude fluctuations of the gravitational field around such solutions;
3. set up a variational calculation of the appropriately defined energy density in the selected background fields;
4. study the possible screening of the unstable modes (if any) of the classical background fields.

As for point (1) we have chosen Schwarzschild's wormhole-solutions [4], the simplest class of solutions of Einstein's equations after flat space-time. In order to achieve (2) the Regge-Wheeler [5] expansion has been systematically employed, yielding two well defined sets of *unstable modes* (for S- and P-wave). This important result, already indicated in previous independent work [6], renders the development of the points (3) and (4) both relevant and meaningful, the former point yielding a lowering of the energy density due to the interaction of the short-wave length modes with the background gravitational field, the latter exhibiting the (approximate) cancellation of the independent components of the Riemann tensor of Schwarzschild's wormholes by the S-wave unstable mode. As a result flat space-time, like the QCD perturbative ground state, becomes "essentially unstable", in the sense that upon it no *stable* quantum dynamics can be realized. On the other hand a well defined "gas of wormholes" appears as a very good candidate for the semi-classical configuration around which the quantized modes of the gravitational field can *stably* fluctuate. But a discussion of the physics implications of our findings must await a more detailed description of our work, which we are now going to provide.

2 The Schrödinger functional approach

In order to develop a functional strategy aiming to determine the Ground State of Quantum Gravity, which parallels the approach developed for QCD [3], we must first identify an appropriate energy functional. In General Relativity, this is a non-trivial problem for, as is well known, in the canonical quantization procedure, first envisaged by Dirac [7] and Arnowitt, Deser and Misner (ADM) [8], due to general covariance the local Hamiltonian is constrained to annihilate the physical ground state, a fact that in the Schrödinger functional approach is expressed by

the celebrated Wheeler-DeWitt equation [9]. However we note that the problem we wish to solve concerns the minimization of the total energy of an "open space", in which there exists a background metric field that becomes "asymptotically flat", i.e. that for spatial infinity ($|\vec{x}| \to \infty$) behaves as

$$g_{ij} \to \delta_{ij} + O(\frac{1}{r}). \tag{1}$$

Thus, we shall have to consider the ADM-energy [8], which in cartesian coordinates is given by ($\partial\Sigma$ is the boundary of the space-region Σ, "$, k$" denotes partial derivative with respect to x_k)

$$E_{ADM} = \frac{1}{16\pi G} \int_{\partial\Sigma} dS^k \delta^{ij} (g_{ik,j} - g_{ij,k}), \tag{2}$$

where

$$g_{ij}(x) = \eta_{ij}(x) + h_{ij}(x), \tag{3}$$

$\eta_{ij}(x)$ being the "asymptotically flat" (see condition (1)) background field, solution of the classical vacuum Einstein's equations. We should like to point out that E_{ADM} is just the energy that an asymptotic observer attributes to space whose time foliation he is keeping anchored to his (asymptotically) flat metric.

We can now expand the total (the sum of the Hamiltonian and E_{ADM}) energy E of space in powers of the quantized fluctuations $h_{ij}(x)$:

$$E = E^{(0)}_{ADM} + E^{(1)}_{ADM} + \sum_n \int_\Sigma d^3x (NH + N_i H^i)^{(n)}, \tag{4}$$

where H and H^i are the super-hamiltonian and super-momentum operators, as defined by ADM [8], and N and N_i are the "lapse-function" and the "shift-vector" of the foliation of the 4-dimensional metric $g_{\mu\nu}$ [8] respectively.

One can easily show that, for a static background[2]

$$E^{(1)}_{ADM} = \frac{1}{16\pi G} \int_{\partial\Sigma} dS^k (h_{kj,j} - h_{jj,k}) = - \int_\Sigma d^3x (NH + N_i H^i)^{(1)}, \tag{5}$$

thus we can rewrite (2) as

$$E = E^{(0)}_{ADM} + \sum_{n \geq 2} \int_\Sigma d^3x (NH + N_i H^i)^{(n)}, \tag{6}$$

The evolution of the physical states $\Psi[h_{ij}]$ with respect to the fixed time of the asymptotic observer is governed by the "Schrödinger equation"

$$i\hbar \partial_t \Psi[h_{ij}] = E \left[h_{kl}, -i\hbar \frac{\delta}{\delta h_{kl}} \right] \Psi[h_{ij}], \tag{7}$$

[2]A detailed account of all the calculations of this letter will be published elsewhere [10].

and it is in this equation, whose Hamiltonian operator is given by (6) and whose wave-functionals obey (order by order in h_{ij}) the superhamiltonian and supermomenta constraints [7, 8], that the parallelism of our problem with the QCD problem [3] is fully regained. Thus in the following we shall look for the minimization of the energy $E^{(2)}$ ($E^{(2)}$ denotes (6) truncated at $n = 2$):

$$E^{(2)} = \int [dh_{ij}]\Psi^*[h_{ij}]E^{(2)}\left[h_{kl}, -i\hbar\frac{\delta}{\delta h_{kl}}\right]\Psi[h_{ij}], \tag{8}$$

on a class of gaussian wave-functionals $\Psi[h_{ij}]$, whose arguments h_{ij} fluctuate around the "wormhole solution" discovered by Schwarzschild in 1916 [7], whose line elements in polar coordinates are given by ($2GM < r < +\infty$)

$$ds^2 = -\frac{r - 2GM}{r}dt^2 + \frac{r}{r - 2GM}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{9}$$

and depend on the single parameter M, the ADM-mass, such that

$$E^{(0)}_{ADM} = M. \tag{10}$$

3 The unstable modes around a single "wormhole": the instability of the Perturbative Ground State

A crucial step in the solution of our problem is the diagonalization of the operator O^{ijkl}, defined by ($N^{(0)} = \sqrt{1 - \frac{2MG}{r}}$)

$$\int_x N^{(0)}V^{(2)} = \frac{1}{16\pi G}\int_x \frac{1}{4N^{(0)}}h_{ij}O^{ijkl}h_{kl}, \tag{11}$$

where ($\eta = \det \eta_{ij}$, η_{ij} being the spatial Schwarzschild metric)

$$\int_x = \int d^3x\sqrt{\eta}, \tag{12}$$

and

$$V^{(2)} = -\frac{1}{16\pi G}(R^{(2)} + \frac{1}{2}h^k_k R^{(1)}), \tag{13}$$

$R^{(1)}, R^{(2)}$ being the first and second order expansions in h_{ij} of the scalar of curvature. Such diagonalization has to be carried out in the space of the wave-functions $h_{ij}(\vec{x})$, subject to the constraints ("$|$" denotes the covariant derivative with respect to the background metric)

$$h^k_k = 0, \tag{14}$$

and

$$\left(\frac{h^{ij}}{N^{(0)}}\right)_{|j} = 0, \tag{15}$$

as demanded by the general covariance.

Calling $\lambda(\rho)$ the eigenvalues of O^{ijkl} (ρ being a complete set of labels), the variational calculation on the gaussian wave-functional [10] shows that the total energy of the system is given by

$$E = M + \frac{\hbar}{2}\sum_{\rho}\sqrt{\lambda(\rho)}, \tag{16}$$

which implies that the PGS, corresponding to $M = 0$, is stable only if all eigenvalues $\lambda(\rho) > 0$. Thus if we find some field modes for which $\lambda(\rho) < 0$, to be called "unstable modes", the implication is that the simple minimization (16) of the second order operator is invalid, requiring the consideration of higher order terms [3]: at present a formidable undertaking. However, the discovery of negative eigenvalues is the unmistakable sign that the PGS is unstable and that Perturbation Theory (PT) is a totally misleading approach to the calculation of the quantum corrections to the free theory (in our case flat space-time). In view of the severe unrenormalizability of the PT of Quantum Gravity this latter fact is certainly good news. Also, as noticed in Ref.[3], the presence of "unstable modes" implies that their contribution to the total energy E is not $O(\hbar)$, as for the "stable modes", but is $O(1)$: a classical contribution, like the ADM-mass M. This means that beyond the quadratic approximation the "classical" contribution from the unstable sector of the h_{ij}-modes will "screen" the ADM-mass by a term $-\epsilon M$ ($0 < \epsilon < 1$). Below we shall argue that the screening is actually complete (i.e. $\epsilon = 1$) like it happens in QCD [3, 11].

A careful analysis [10] of the eigenvalue problem posed by the operator O^{ijkl} in the space of eigenmodes obeying (14, 15) reveals that around a single "wormhole" there exist at least four unstable modes, one in S-wave and three in P-wave with degenerate eigenvalue

$$\lambda = -\frac{1}{16(GM)^2}, \tag{17}$$

The contribution of the stable modes ($\lambda(\rho)$ positive) to the energy (16) can be easily evaluated through a WKB-analysis [10], that shows it to be smaller than the zero-point graviton contribution to the PGS energy. This important result is due to the red-shift experienced by these modes in the gravitational field of the "wormhole". Defining,

$$\Delta E(M) = E(M) - E(0), \tag{18}$$

$E(0)$ being the PGS energy in the quadratic approximation, we can finally write (Λ is the ultraviolet cutoff)

$$\Delta E(M) = M(1-\epsilon) - \frac{64\Lambda^4}{\pi^3}R^2 GM\ell n\left(\frac{R}{2GM}\right), \tag{19}$$

where R ($R \gg 2GM$) is the radius of the spherical region that surrounds the "wormhole" where the space metric differs appreciably from the metric of the asymptotic observer. From (18) one can obtain an "average" energy density through simple division by the volume $\frac{4}{3}\pi R^3$ of the spherical region. As expected the quantum corrections to the classical "wormhole" energy $M(1-\epsilon)$ (which comprises the screening of the "unstable modes") turns out to be negative.

Following a general idea due to R. Feynman [11] we shall now argue that $\epsilon = 1$, i.e. the "screening" of the unstable modes is total. Indeed, from the explicit solutions of the "unstable modes" we may compute their contribution to the Riemann tensor up to a possible multiplicative factor, A, the square of their (unknown) amplitudes [10]. Averaging the components of the classical part of the full Riemann tensor over the spherical region of radius R, we find an approximately vanishing value for the "average" classical Riemann tensor, implying that the asymptotic observer will perceive the classical configuration of the metric field inside the spherical region as (approximately) flat.

In this way flat space-time is seen to be quantum mechanically unstable, the energy density of space around a wormhole of extension $r_{WH} = 2GM$ differring from it by the quantity

$$\Delta E(M,R) \simeq -\frac{48}{\pi^4}\Lambda^4 \frac{GM}{R} \ell n \left(\frac{R}{2GM}\right). \tag{20}$$

The question now is what is the "multi-wormhole" configuration that minimizes the energy density of the whole space? We shall try to give a plausible answer to this question in the next Section.

4 The "wormhole gas", the possible GS of QG and the Planck lattice

The analysis we have carried out so far has yielded the expression (20) for the average energy density gain (with respect to flat space-time) of a spherical region of radius R around a "wormhole" of size $2GM$. In order to minimize the energy of the quantum gravitational field in the whole space it seems natural to consider a collection of such regions filling space completely. We are thus led to look at a "gas of wormholes" with density $\sim \frac{1}{R^3}$ and size $2GM \leq R$ as the (quasi)-classical field configuration around which the gravitational quantum field fluctuates: our model for the Ground State of Quantum Gravity.

It is important to note that a "gas of wormholes" cannot form a stable background field for the quantized gravitational field unless the minimum distance between any two of them, due to their gravitational interaction, is bigger than their size $2GM$. This observation clearly implies a constraint on the ADM-mass M. Let us see how. Following the interesting work of Gibbons and Perry [12] we

know that the interaction potential between two wormholes of ADM-mass M is just the Newtonian one

$$U(d) = -\frac{GM^2}{d}, \tag{21}$$

d being the distance of their centers. Thus in classical physics there is nothing to prevent their collapse into a single wormhole of appropriate ADM-mass $M' < 2M$. This situation of basic classical instability changes when we treat the wormholes as quantum particles of mass M and size $2GM$, as appropriate to their physical nature as perceived by the asymptotic observer. The ensuing quantum mechanical problem, akin to the paradigmatic hydrogen atom problem, yields (for the rest of this paper $\hbar = c = 1$)

$$r_\circ = \frac{2}{GM^3} \tag{22}$$

for the "Bohr-radius" of the quantum mechanical ground state, and

$$E_W = -\frac{1}{2}\frac{GM^2}{r_\circ} = -\frac{G^2M^5}{4} \tag{23}$$

for the binding energy of the two-wormholes system. Thus in order to have a well defined and stable gas of wormholes we must have

$$r_\circ = \frac{2}{GM^3} > 2GM \tag{24}$$

i.e.

$$M < G^{-\frac{1}{2}} = m_p, \tag{25}$$

stating that the mass of the wormholes of the gas must be smaller than the Planck-mass m_p. A very important finding.

We are now in a position to determine the actual size R of the spherical domains and of the wormholes $GM < G^{\frac{1}{2}}$ (the Planck length, a_p) of the Ground state of Quantum Gravity. Indeed, in order to achieve this we only need minimize the energy density ($R = 2\lambda GM$)

$$\Delta E(M,\lambda) = -\frac{24\Lambda^4}{\pi^4\lambda}\ell n\lambda - \frac{3}{256\pi}\frac{M^2}{G\lambda^3}, \tag{26}$$

where the last term stems from the binding energy between pairs of wormholes. The result of the minimization is

$$\lambda \simeq e, \tag{27}$$

and

$$M = G^{-\frac{1}{2}} = m_p, \tag{28}$$

the extreme of the allowed range for the ADM-mass M (see eq.(25)).

The above chain of deductions delivers us a state of the quantized gravitational field whose energy density (as seen by an asymptotic observer living over a flat space-time) is vastly $[O(\Lambda^4)]$ below the energy density of the Perturbative Ground State. This state consists of a (semi)-classical background whose structure is that of an interacting gas of wormholes whose average distance is of the order of the Planck length $a_p = G^{\frac{1}{2}}$, and its average ADM-mass the Planck mass $m_p = G^{-\frac{1}{2}}$. Recalling that for the outside observer the radius $2GM$ of the wormhole is an unpenetrable horizon, the space-configuration of this putative GS is equivalent to a random lattice with average lattice constant a_p. Indeed the only reasonable and meaningful description our observer can make of this peculiar configuration is to assign an average value of the metric field to the spatial interstices existing among the voids created by the wormholes. An adequate approximate picture of space-time that captures the essential features of such state is then that of a regular lattice with the lattice constant a_p: the Planck lattice.

5 Conclusion

In this letter we have tried to gain some understanding of the structure of the GS of Einstein's Quantum Gravity. In order to achieve this ambitious goal we have focussed our attention on the gauge-structure of Einstein's General Relativity (GR) and envisaged a strategy of analysis that for a non-abelian gauge theory, such as QCD, had proved deeply insightful [3]. We have thus tried out a variational calculation of the appropriately defined energy functional of the quantum fluctuations of the gravitational field around the classical solution of matterless GR: Schwarzschild's wormholes. Having found negative eigenvalues (associated to "unstable modes") for the second order wave-operator arising from the lowest order expansion of the GR Hamiltonian, we have given arguments for the cancellation of the classical $[O(1)]$ term of the energy, receiving contribution from the wormhole and the S-wave "unstable mode". Deprived of the positive classical energy term the energy density of the quantum gravitational field around a wormhole turns out to be lower than that of the quantum fluctuations over flat space-time, the Perturbative Ground State, thus explicitly proving that the PGS of QG and the awfully non-renormalizable perturbation theory are physically meaningless, due to the "essential" [3] instability of the PGS.

By extending the single wormhole configuration to a multi-wormhole one, we have discovered that the size $2GM$ of the wormholes is quantum mechanically limited by the Planck length $a_p = G^{\frac{1}{2}}$ and, consequently, the ADM-mass by the Planck mass $m_p = G^{-\frac{1}{2}}$. As a result in the the state of minimum energy space is seen to acquire the structure of a random lattice with average lattice constant a_p. A space-structure that can be adequately modeled by a Planck lattice.

Before discussing the physics conclusions that one may draw from such a characteristic structure of the putative GS of QG, we would like to warn the reader

about the still speculative, though eminently plausible nature of our results. The only thing of this work that definitely stands on solid ground, in fact, is the discovery of the instability of the PGS of Quantum Gravity, and of its unavoidable irrelevance for the quantum dynamics of the gravitational field. We deem this fact quite important, for it clears the way of QG from the embarassments of a disastrously unrenormalizable theoretical structure, that only belongs, as far as we know, to an approach, PT, whose very foundations are thus seen to dematerialize. Having so disposed of the *pars destruens*, what can we say about the *pars construens* of this paper?

If we accept that wormholes are an essential ingredient not of a local minimum of the energy density in the configurational space of the quantum gravitational field, but of the absolute minimum, the true ground state, then a satisfactory solution appears in sight of the long standing problem of local Quantum Field Theories (QFT's), that so upset among others Dirac: the infinities that the renormalization program cleverly manipulates but does not really eliminate from the theory. Indeed, if the small scale structure of space-time is well described by a Planck lattice of lattice constant a_p, then for the QFT's of the Standard Model there exists a natural cutoff at the Planck mass m_p, and the logarithmic infinities of those "renormalizable" continuous QFT's become now *small corrections*, as appropriate for terms of a perturbative series. A most welcome result.

We find in this plausible derivation (modulo the *caveats* already expressed) of Wheeler's Quantum Foam, that realizes the prophetical ideas of Riemann's "Über die Hypothesen welche der Geometrie zu Grunde liegen" [13], the most pleasing and possibly fertile outcome of this work.

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